

Hamiltonian Formalism of Two-Dimensional Vlasov Kinetic Equation

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Abstract

In this paper the two-dimensional Benney system describing long wave propagation of a finite depth fluid motion and the multi-dimensional Russo–Smereka kinetic equation describing a bubbly flow are considered. The Hamiltonian approach established by J. Gibbons for one-dimensional Vlasov kinetic equation is extended to a multi-dimensional case. A local Hamiltonian structure associated with the hydrodynamic lattice of moments derived by D.J. Benney is constructed. A relationship between this hydrodynamic lattice of moments and the two-dimensional Vlasov kinetic equation is found. In the two-dimensional case a Hamiltonian hydrodynamic lattice for the Russo–Smereka kinetic model is constructed. Simple hydrodynamic reductions are presented.

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keywords: collisionless kinetic equation, Hamiltonian structure, hydrodynamic lattice, hydrodynamic reduction.

1 Introduction

Recently (see [2]) D.J. Benney considered a three-dimensional motion of a finite depth fluid. Corresponding nonlinear system in partial derivatives is written on three components of the velocity¹ $u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)$ and the profile $\eta(x, z, t)$ of a free surface²:

$$\begin{aligned} u_x + v_y + w_z &= 0, \\ u_t + uu_x + vu_y + wu_z &= -\eta_x, \\ w_t + uw_x + vw_y + ww_z &= -\eta_z \end{aligned} \quad (1)$$

with the boundary conditions

$$\begin{aligned} v &= 0, \quad y = 0, \\ \eta_t + u\eta_x + w\eta_z - v &= 0, \quad y = \eta. \end{aligned}$$

Introducing infinitely many moments

$$A^{k,m}(x, z, t) = \int_0^\eta u^k(x, y, z, t) w^m(x, y, z, t) dy, \quad (2)$$

D.J. Benney derived a two-dimensional hydrodynamic chain or we shall call a “*hydrodynamic lattice*”

$$A_t^{k,m} + A_x^{k+1,m} + A_z^{k,m+1} + kA^{k-1,m}A_x^{0,0} + mA^{k,m-1}A_z^{0,0} = 0, \quad k, m = 0, 1, \dots, \quad (3)$$

which has just four local conservation laws, i.e.

$$\begin{aligned} A_t^{0,0} + A_x^{1,0} + A_z^{0,1} &= 0, \\ A_t^{1,0} + \left(A^{2,0} + \frac{1}{2}(A^{0,0})^2 \right)_x + A_z^{1,1} &= 0, \\ A_t^{0,1} + A_x^{1,1} + \left(A^{0,2} + \frac{1}{2}(A^{0,0})^2 \right)_z &= 0, \\ (A^{0,2} + A^{2,0} + (A^{0,0})^2)_t + (A^{3,0} + A^{1,2} + 2A^{1,0}A^{0,0})_x + (A^{2,1} + A^{0,3} + 2A^{0,1}A^{0,0})_z &= 0. \end{aligned}$$

In next Section of this paper we present a local Hamiltonian structure³ of (3)

$$A_t^{i,j} + [iA^{i+k-1,j+m}\partial_x + k\partial_x A^{i+k-1,j+m} + jA^{i+k,j+m-1}\partial_z + m\partial_z A^{i+k,j+m-1}] \frac{\delta \mathbf{H}}{\delta A^{k,m}} = 0, \quad (4)$$

¹according to the notation introduced by D.J. Benney, x and z are horizontal Cartesian coordinates, y is a vertical Cartesian coordinate; u and w are horizontal components of the velocity, v is a vertical component of the velocity.

²any lower index means a corresponding derivative.

³we assume summations with respect to each repeated index here and everywhere below; any pure differential operator $\partial_x, \partial_z, \partial_p, \partial_q$ acts on all that is written in r.h.s.

consider simplest hydrodynamic reductions of this Benney hydrodynamic lattice and construct a relationship with the two-dimensional Vlasov (collisionless Boltzmann) kinetic equation (see, for instance, [15])

$$f_t + pf_x + qf_z - f_p A_x^{0,0} - f_q A_z^{0,0} = 0. \quad (5)$$

Third Section of this paper is devoted to another two-dimensional Vlasov kinetic equation appearing in description of a bubbly flow (see [13]). Corresponding Hamiltonian hydrodynamic lattice also is found. Simplest three-dimensional hydrodynamic reductions for both hydrodynamic lattices are extracted.

2 The Benney System

In this paper we utilize the approach established in [7] and later significantly developed in [8] for a one-dimensional case. *We show that this approach is useful also for a **multi-dimensional** case.* Without loss of generality and for simplicity we restrict our consideration to two-dimensional case only.

1. Introducing moments⁴

$$A^{k,m}(x, z, t) = \int \int p^k q^m f(x, z, p, q, t) dp dq, \quad k, m = 0, 1, \dots \quad (6)$$

Vlasov kinetic equation (5) implies Benney hydrodynamic lattice (3).

2. Vlasov kinetic equation (5) possesses the Hamiltonian structure

$$f_t = \{f, H\}_{\text{LP}}, \quad (7)$$

where the standard (ultralocal) Lie–Poisson bracket

$$\{f, g\}_{\text{LP}} := f_p g_x + f_q g_z - f_x g_p - f_z g_q,$$

and the Hamiltonian function $H = \frac{1}{2}(p^2 + q^2) + A^{0,0}$.

3. Introducing the functional

$$\mathbf{H} = \frac{1}{2} \int \int (A^{0,2} + A^{2,0} + (A^{0,0})^2) dx dz \quad (8)$$

such that $H = \delta \mathbf{H} / \delta f$, Vlasov kinetic equation (5) also can be written in the theoretic-field Hamiltonian form (cf. (7))

$$f_t = \{f, \mathbf{H}\}, \quad (9)$$

where

$$\{f, \mathbf{H}\} := (f_p \partial_x + f_q \partial_z - f_x \partial_p - f_z \partial_q) \frac{\delta \mathbf{H}}{\delta f}.$$

Indeed,

$$\frac{\delta \mathbf{H}}{\delta f} = \frac{1}{2} \frac{\delta A^{0,2}}{\delta f} + \frac{1}{2} \frac{\delta A^{2,0}}{\delta f} + A^{0,0} \frac{\delta A^{0,0}}{\delta f} = \frac{1}{2}(p^2 + q^2) + A^{0,0} \equiv H,$$

⁴these integrals converge, for example, if λ is bounded and $|\lambda| \rightarrow 0$ faster than any $(\Sigma |p_s|)^{-K}$, $\forall K \geq 1$.

where we utilized the elementary variational property (see (6))

$$\frac{\delta A^{k,m}}{\delta f} = p^k q^m.$$

Thus Vlasov kinetic equation (5) is an integro-differential equation (see (6)):

$$f_t + pf_x + qf_z - f_p A_x^{0,0} - f_q A_z^{0,0} = 0, \quad A^{0,0}(x, z, t) = \int \int f(x, z, p, q, t) dp dq. \quad (10)$$

4. The relation between Vlasov kinetic equation (5) written in Hamiltonian form (9) and Benney hydrodynamic lattice (3) written in Hamiltonian form (4) is obtained by defining mapping

$$\mu : f(x, z, p, q, t) \longmapsto \{A^{k,m}(x, z, t)\}_{k,m=0,1,\dots}^\infty,$$

where moments $A^{k,m}$ are determined by (6). Indeed, in a general case (cf. (8))

$$\mathbf{H} = \int \int h(A^{0,0}, A^{0,1}, A^{1,0}, A^{0,2}, A^{1,1}, A^{2,0}, \dots) dx dz \quad (11)$$

and

$$\frac{\delta \mathbf{H}}{\delta f} = \frac{\partial h}{\partial A^{k,m}} \frac{\delta A^{k,m}}{\delta f} = p^k q^m \frac{\partial h}{\partial A^{k,m}}.$$

Then multiplying (9) by $p^k q^m$ and integrating over p and q (see (6)),

$$\begin{aligned} A_t^{i,j} &= \int \int p^i q^j f_t dp dq = \int \int p^i q^j \{f, \mathbf{H}\} dp dq \\ &= \int \int p^i q^j \left[f_p \left(\frac{\delta \mathbf{H}}{\delta f} \right)_x + f_q \left(\frac{\delta \mathbf{H}}{\delta f} \right)_z - f_x \left(\frac{\delta \mathbf{H}}{\delta f} \right)_p - f_z \left(\frac{\delta \mathbf{H}}{\delta f} \right)_q \right] dp dq \\ &= \left(\frac{\partial h}{\partial A^{k,m}} \right)_x \int \int p^{k+i} q^{m+j} f_p dp dq + \left(\frac{\partial h}{\partial A^{k,m}} \right)_z \int \int p^{k+i} q^{m+j} f_q dp dq \\ &\quad - k \frac{\partial h}{\partial A^{k,m}} \int \int p^{k+i-1} q^{m+j} f_x dp dq - m \frac{\partial h}{\partial A^{k,m}} \int \int p^{k+i} q^{m+j-1} f_z dp dq, \end{aligned}$$

then integrating by parts two first integrals (with respect to p and q , respectively), we obtain

$$\begin{aligned} &= -(k+i) A^{k+i-1, m+j} \left(\frac{\partial h}{\partial A^{k,m}} \right)_x - (m+j) A^{k+i, m+j-1} \left(\frac{\partial h}{\partial A^{k,m}} \right)_z \\ &\quad - k \frac{\partial h}{\partial A^{k,m}} A_x^{k+i-1, m+j} - m \frac{\partial h}{\partial A^{k,m}} A_z^{k+i, m+j-1}. \end{aligned}$$

This is nothing but precisely (4), where obviously (see (11))

$$\frac{\delta \mathbf{H}}{\delta A^{k,m}} \equiv \frac{\partial h}{\partial A^{k,m}}.$$

In particular case (8), Hamiltonian system (4) yields Benney hydrodynamic lattice (3). Corresponding two-dimensional Poisson bracket

$$\begin{aligned} \{A^{i,j}(x, z), A^{k,m}(x', z')\} = & [(i+k)A^{i+k-1,j+m}\partial_x + kA_x^{i+k-1,j+m} \\ & + (j+m)A^{i+k,j+m-1}\partial_z + mA_z^{i+k,j+m-1}]\delta(x-x')\delta(z-z') \end{aligned} \quad (12)$$

we call the *two-dimensional* Kupershmidt–Manin Poisson bracket, because earlier (see [9]) the *one-dimensional* Kupershmidt–Manin Poisson bracket

$$\{A^k(x), A^m(x')\} = [(k+m)A^{k+m-1}\partial_x + mA_x^{k+m-1}]\delta(x-x') \quad (13)$$

was derived for the one-dimensional Benney hydrodynamic chain (see [1])

$$A_t^k + A_x^{k+1} + kA^{k-1}A_x^0 = 0, \quad k = 0, 1, \dots$$

Poisson bracket (12) is a *first* example of *two-dimensional* differential-geometric Poisson brackets of a first order (see [4], [10] and [5]) generalised to an *infinitely* many component case.

In the previous Section we mentioned that D.J. Benney was able to find just four local conservation laws. Actually, explanation of this result is based on existence of local Hamiltonian structure (4). Indeed, any Hamiltonian system (4) possesses just four local conservation laws for an arbitrary Hamiltonian density (11):

1. the continuity conservation law

$$A_t^{0,0} + \left(kA^{k-1,m} \frac{\partial h}{\partial A^{k,m}} \right)_x + \left(mA^{k,m-1} \frac{\partial h}{\partial A^{k,m}} \right)_z = 0;$$

2. the conservation law of the momentum (x, z components)

$$A_t^{1,0} + \left[(k+1)A^{k,m} \frac{\partial h}{\partial A^{k,m}} - h \right]_x + \left(mA^{k+1,m-1} \frac{\partial h}{\partial A^{k,m}} \right)_z = 0,$$

$$A_t^{0,1} + \left(kA^{k-1,m+1} \frac{\partial h}{\partial A^{k,m}} \right)_x + \left[(m+1)A^{k,m} \frac{\partial h}{\partial A^{k,m}} - h \right]_z = 0;$$

3. the conservation law of the energy

$$h_t + \left(kA^{i+k-1,j+m} \frac{\partial h}{\partial A^{i,j}} \frac{\partial h}{\partial A^{k,m}} \right)_x + \left(mA^{i+k,j+m-1} \frac{\partial h}{\partial A^{i,j}} \frac{\partial h}{\partial A^{k,m}} \right)_z = 0.$$

Any other local conservation laws can exist just for very special dependencies of Hamiltonian density $h(A^{0,0}, A^{0,1}, A^{1,0}, A^{0,2}, A^{1,1}, A^{2,0}, \dots)$. We believe that corresponding hydrodynamic lattices must be integrable. This problem should be considered in a separate paper.

The construction presented in this Section is so natural, that without any restrictions, one can generalise two-dimensional Kupershmidt–Manin Poisson bracket (12) to any higher number of spatial dimensions.

3 The Russo–Smereka Model

The Russo–Smereka model (see [13]) describing a bubbly flow contains the Vlasov kinetic equation

$$f_t + u^k \frac{\partial f}{\partial x^k} - \frac{\partial(p_m u^m)}{\partial x^k} \frac{\partial f}{\partial p_k} = 0, \quad (14)$$

where $(\rho_l$ and τ are parameters, indices k, m run from 1 up to n)

$$\rho_l u^k = \frac{2}{\tau} p_k + 3(j_k - 3\partial_k \Phi), \quad j_k = \int p_k f d\mathbf{p}$$

and the Poisson equation

$$\partial_m^2 \Phi = \partial_m j_m. \quad (15)$$

In this Section we adopt the approach established in [7] and developed in [8] to the case equipped by extra constraints.

Theorem: *Vlasov kinetic equation (14) and Poisson equation (15) can be written in the theoretic-field Hamiltonian form (cf. (9))*

$$f_t = \{f, \mathbf{H}\}, \quad \frac{\delta \mathbf{H}}{\delta \Phi} = 0, \quad (16)$$

where the Poisson bracket is

$$\{f, \mathbf{H}\} := \left(\frac{\partial f}{\partial p_m} \frac{\partial}{\partial x^m} - \frac{\partial f}{\partial x^m} \frac{\partial}{\partial p_m} \right) \frac{\delta \mathbf{H}}{\delta f}$$

and the Hamiltonian is

$$\mathbf{H} = \frac{9}{2\rho_l} \sum_{k=1}^n \int \left(\frac{2}{9\tau} \int p_k^2 f d\mathbf{p} + \frac{1}{3} j_k^2 - 2j_k \partial_k \Phi + (\partial_k \Phi)^2 \right) d\mathbf{x}. \quad (17)$$

Proof: Substitution Hamiltonian (17) into (16) yields (14) and (15), respectively.

Without loss of generality and for simplicity here we again restrict our consideration on two-dimensional case only. Then Hamiltonian system (16)

$$f_t = (f_p \partial_x + f_q \partial_z - f_x \partial_p - f_z \partial_q) \frac{\delta \mathbf{H}}{\delta f}, \quad \frac{\delta \mathbf{H}}{\delta \Phi} = 0$$

is determined by the Hamiltonian (17)

$$\mathbf{H} = \frac{9}{2\rho_l} \int \left(\frac{2}{9\tau} (A^{0,2} + A^{2,0}) + \frac{1}{3} [(A^{0,1})^2 + (A^{1,0})^2] - 2(A^{1,0} \Phi_x + A^{0,1} \Phi_z) + \Phi_x^2 + \Phi_z^2 \right) dx dz.$$

Corresponding Hamiltonian hydrodynamic lattice (4) takes the form (cf. (3); here both parameters ρ_l and τ removed by appropriate scaling of independent variables x, z, t and dependent functions $A^{k,m}$ and Φ)

$$0 = A_t^{i,j} + A_z^{i,j+1} + A_x^{i+1,j} \quad (18)$$

$+[iA^{i-1,j+1}\partial_x+(j+1)A^{i,j}\partial_z+A_z^{i,j}](A^{0,1}-3\Phi_z)+[(i+1)A^{i,j}\partial_x+A_x^{i,j}+jA^{i+1,j-1}\partial_z](A^{1,0}-3\Phi_x)$,
where Poisson equation (15) becomes

$$\Phi_{xx} + \Phi_{zz} = A_x^{1,0} + A_z^{0,1}.$$

This constraint $\delta\mathbf{H}/\delta\Phi = 0$ can be written in the obvious conservative form $\partial_m(\partial_m\Phi - j_m) = 0$. Thus in the two-dimensional case one extra conservation law exists, i.e.

$$(\Phi_x - A^{1,0})_x + (\Phi_z - A^{0,1})_z = 0, \quad (19)$$

while the hydrodynamic lattice possesses just four local conservation laws (cf. two previous Sections):

1. the continuity conservation law

$$A_t^{0,0} + (A^{1,0} + A^{0,0}A^{1,0} - 3A^{0,0}\Phi_x)_x + (A^{0,1} + A^{0,0}A^{0,1} - 3A^{0,0}\Phi_z)_z = 0,$$

2. the conservation law of the momentum (x, z components)

$$\begin{aligned} & A_t^{0,1} + (A^{1,1} + A^{0,1}A^{1,0} - 3A^{0,1}\Phi_x - 3A^{1,0}\Phi_z + 3\Phi_x\Phi_z)_x \\ & + \left(A^{0,2} + \frac{3}{2}(A^{0,1})^2 + \frac{1}{2}(A^{1,0})^2 - 6A^{0,1}\Phi_z + \frac{3}{2}\Phi_z^2 - \frac{3}{2}\Phi_x^2 \right)_z = 0, \\ & A_t^{1,0} + (A^{1,1} + A^{0,1}A^{1,0} - 3A^{0,1}\Phi_x - 3A^{1,0}\Phi_z + 3\Phi_x\Phi_z)_z \\ & + \left(A^{2,0} + \frac{1}{2}(A^{0,1})^2 + \frac{3}{2}(A^{1,0})^2 - 6A^{1,0}\Phi_x + \frac{3}{2}\Phi_x^2 - \frac{3}{2}\Phi_z^2 \right)_x = 0, \end{aligned}$$

3. the conservation law of the energy

$$\begin{aligned} 0 = & (A^{0,2} + A^{2,0} + (A^{0,1})^2 + (A^{1,0})^2 - 6A^{1,0}\Phi_x - 6A^{0,1}\Phi_z + 3\Phi_x^2 + 3\Phi_z^2)_t \\ & + (A^{3,0} + A^{1,2} + A^{1,0}A^{0,2} + 3A^{2,0}A^{1,0} - 9A^{2,0}\Phi_x - 3A^{0,2}\Phi_x + 2A^{0,1}A^{1,1} + 2(A^{1,0})^3 + 2(A^{0,1})^2A^{1,0} \\ & + 18A^{1,0}\Phi_x^2 + 18A^{0,1}\Phi_x\Phi_z + 6(A^{1,0} - \Phi_x)\Phi_t - 6(A^{1,1} + A^{0,1}A^{1,0})\Phi_z - 6[(A^{0,1})^2 + 2(A^{1,0})^2]\Phi_x)_x \\ & + (A^{2,1} + A^{0,3} + A^{0,1}A^{2,0} + 3A^{0,2}A^{0,1} - 9A^{0,2}\Phi_z - 3A^{2,0}\Phi_z + 2A^{1,0}A^{1,1} + 2(A^{0,1})^3 + 2(A^{1,0})^2A^{0,1} \\ & + 18A^{0,1}\Phi_z^2 + 18A^{1,0}\Phi_x\Phi_z + 6(A^{0,1} - \Phi_z)\Phi_t - 6(A^{1,1} + A^{0,1}A^{1,0})\Phi_x - 6[(A^{1,0})^2 + 2(A^{0,1})^2]\Phi_z)_z. \end{aligned}$$

4 Multi-Component Reductions

Investigation of infinitely many component quasilinear systems of a first order (i.e. hydrodynamic lattices) is very complicated problem. One of most effective tools is a method of multi-component hydrodynamic reductions. In a general case hydrodynamic reductions can be extracted utilizing *generalized* functions (see, for instance, [3]) like the Heaviside step-function or the Dirac delta-function.

For example, the so called “cold plasma” approximation ansatz⁵ (see, for instance, [14])

$$f(x, z, p, q, t) = \sum_{k=1}^N \eta^k(x, z, t) \delta(p - u^k(x, z, t)) \delta(q - w^k(x, z, t))$$

reduces Benney hydrodynamic lattice (3) to the finite component form

$$\eta_t^k + (u^k \eta^k)_x + (w^k \eta^k)_z = 0,$$

$$u_t^k + u^k u_x^k + w^k u_z^k + \left(\sum_{m=1}^N \eta^m \right)_x = 0, \quad w_t^k + u^k w_x^k + w^k w_z^k + \left(\sum_{m=1}^N \eta^m \right)_z = 0,$$

where all moments are expressed polynomially via new field variables η^k, u^s, w^m (see (6) and more detail in [11])

$$A^{k,m} = \sum_{p=1}^N \eta^p (u^p)^k (w^p)^m. \quad (20)$$

In the simplest case $N = 1$, this is nothing but a system describing the irrotational two-dimensional hydrodynamics

$$\eta_t + (u\eta)_x + (w\eta)_z = 0, \quad (21)$$

$$u_t + uu_x + wu_z + \eta_x = 0, \quad w_t + uw_x + ww_z + \eta_z = 0.$$

1. This system also can be obtained directly from two-dimensional Benney system (1) by the reduction⁶ $u(x, z, t), w(x, z, t)$ and $v = -y(u_x + w_z)$.

2. This system possesses a local Hamiltonian structure (see, for instance, [12]), where corresponding Poisson bracket is (here we write just nonzero components)

$$\{\eta(x, z), u(x', z')\} = \{u(x, z), \eta(x', z')\} = \delta'(x - x') \delta(z - z'),$$

$$\{\eta(x, z), w(x, z)\} = \{w(x, z), \eta(x', z')\} = \delta(x - x') \delta'(z - z'),$$

$$\{u(x, z), w(x', z')\} = -\{w(x, z), u(x', z')\} = \frac{u_z - w_x}{\eta} \delta(x - x') \delta(z - z').$$

Indeed, the substitution $A^{k,m} = \eta u^k w^m$ into two-dimensional Kupershmidt–Manin Poisson bracket (12) implies above two-dimensional Poisson bracket.

3. This system possesses just four local conservation laws (see the end of the previous Section)

$$\eta_t + (u\eta)_x + (w\eta)_z = 0,$$

$$(u\eta)_t + \left(u^2 \eta + \frac{1}{2} \eta^2 \right)_x + (uw\eta)_z = 0,$$

$$(w\eta)_t + (uw\eta)_x + \left(w^2 \eta + \frac{1}{2} \eta^2 \right)_z = 0,$$

⁵also well known as the “multi-flow” ansatz.

⁶substitution reduced dependencies $u(x, z, t)$ and $w(x, z, t)$ into (2) transforms hydrodynamic lattice (3) to above three component two dimensional hydrodynamic type system (21).

$$[(u^2 + w^2)\eta + \eta^2]_t + [(u^2 + w^2)u\eta + 2u\eta^2]_x + [(u^2 + w^2)w\eta + 2w\eta^2]_z = 0.$$

In comparison with hydrodynamic lattice (3) the substitution (20) into hydrodynamic lattice (18) leads to another multi-component three-dimensional quasilinear system of a first order⁷ (here we introduced two extra field variables $a = \Phi_x$ and $b = \Phi_z$)

$$\begin{aligned} \eta_t^k + [\eta^k(A^{1,0} - 3a + u^k)]_x + [\eta^k(A^{0,1} - 3b + w^k)]_z &= 0, \\ u_t^k + u^k u_x^k + w^k u_z^k + [w^k \partial_x + u_z^k](A^{0,1} - 3b) + [u^k(A^{1,0} - 3a)]_x &= 0, \\ w_t^k + w^k w_z^k + u^k w_x^k + [w^k(A^{0,1} - 3b)]_z + [w_x^k + u^k \partial_z](A^{1,0} - 3a) &= 0, \\ a_z = b_x, \quad (a - A^{1,0})_x + (b - A^{0,1})_z &= 0, \end{aligned}$$

where two last equations follow from the constraint $\delta \mathbf{H} / \delta \Phi = 0$ (see (19)).

Obviously any solutions of above finite-component quasilinear systems of a first order are simultaneously solutions of hydrodynamic lattices (3), (18). Construction of these solutions should be made in a separate paper.

5 Conclusion

In this paper we extended the Hamiltonian approach established in [7] and developed in [8] to a multi-dimensional case and to the case equipped by extra constraints, which can be written in a variational form. We constructed Hamiltonian structure (4) for Benney hydrodynamic lattice (3) associated with two-dimensional Benney system (1) and we established a link between two-dimensional Benney system (1) and two-dimensional Vlasov kinetic equation (5), which is an integro-differential equation (10). We constructed hydrodynamic lattice (18) associated with two-dimensional Russo–Smereka kinetic equation (see (14)), and we found a Hamiltonian structure of this hydrodynamic lattice. Also, simple three-dimensional hydrodynamic reductions are presented.

Here we just mention how the construction presented in this paper can be utilized. If, for instance, instead of the Hamiltonian density (see (8))

$$h = \frac{1}{2}[A^{0,2} + A^{2,0} + (A^{0,0})^2]$$

to substitute a slightly more general expression (here $Q(a)$ is an arbitrary function)

$$h = \frac{1}{2}[A^{0,2} + A^{2,0} + Q(A^{0,0})]$$

into (4), one can obtain the hydrodynamic lattice

$$A_t^{k,m} + A_x^{k+1,m} + A_z^{k,m+1} + kA^{k-1,m}Q''(A^{0,0})A_x^{0,0} + mA^{k,m-1}Q''(A^{0,0})A_z^{0,0} = 0, \quad k, m = 0, 1, \dots,$$

whose three component hydrodynamic reduction determined by the moment decomposition $A^{k,m} = \eta u^k w^m$ is the nonlinear system describing irrotational two-dimensional barotropic hydrodynamics (cf. (21))

$$\eta_t + (u\eta)_x + (w\eta)_z = 0,$$

⁷also first two equations of this quasilinear system are nothing but the Poisson equation written in conservative form (19).

$$u_t + uu_x + wu_z + Q''(\eta)\eta_x = 0, \quad w_t + uw_x + ww_z + Q''(\eta)\eta_z = 0.$$

Thus this nonlinear system (as in the previous Section) is the reduction $u(x, z, t)$, $w(x, z, t)$ and $v = -y(u_x + w_z)$ of the barotropic “Benney type” fluid (cf. (1))

$$u_x + v_y + w_z = 0,$$

$$u_t + uu_x + vu_y + wu_z + Q''(\eta)\eta_x = 0,$$

$$w_t + uw_x + vw_y + ww_z + Q''(\eta)\eta_z = 0$$

with the boundary conditions

$$v = 0, \quad y = 0,$$

$$\eta_t + u\eta_x + w\eta_z - v = 0, \quad y = \eta.$$

More complicated Hamiltonian hydrodynamic lattices and more complicated hydrodynamic reductions (see, for instance, [3]) should be investigated in a separate paper.

We hope that two-dimensional Kupershmidt–Manin Poisson bracket (12) will play the same important role in the theory of integrable hydrodynamic lattices as well as one-dimensional Kupershmidt–Manin Poisson bracket (13) in the theory of integrable hydrodynamic chains (see, for instance, [11] and [6]).

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